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Camassa–Holm, Korteweg–de Vries-5 and other asymptotically equivalent equations for shallow water waves

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Dedicated to Phil Drazin, remembering his great wit and kindness

Abstract

We derive the Camassa–Holm equation (CH) as a shallow water wave equation with surface tension in an asymptotic expansion that extends one order beyond the Korteweg–de Vries equation (KdV). We show that CH is asymptotically equivalent to KdV5 (the fifth-order integrable equation in the KdV hierarchy) by using the non-linear/non-local transformations introduced in Kodama (Phys. Lett. A 107 (1985a) 245; Phys. Lett. A 112 (1985b) 193; Phys. Lett. A 123 (1987) 276). We also classify its travelling wave solutions as a function of Bond number by using phase plane analysis. Finally, we discuss the experimental observability of the CH solutions.

1. Introduction

We study the irrotational incompressible flow of a shallow layer of inviscid fluid moving under the influence of gravity as well as surface tension. Previously Dullin et al. (2001) studied the case without surface tension, which in the shallow water approximation leads to the Camassa–Holm equation (CH). This is the following 1+1 quadratically non-linear equation for unidirectional water waves with fluid velocity $u(x,t)$,

$$m_t + c_0 m_x + u m_x + 2m u_x + \Gamma u_{xxx} = 0. \tag{1.1}$$

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Here $m=u-\alpha^2u_{xx}$ is a momentum variable, partial derivatives are denoted by subscripts, the constants α^2 and Γ/c_0 are squares of length scales and $c_0=\sqrt{gh}$ is the linear wave speed for undisturbed water at rest at spatial infinity, where u and m are taken to vanish. Setting $\alpha^2\to 0$ in (1.1) recovers the Korteweg–de Vries (KdV) equation of Korteweg and de Vries (1895).

Eq. (1.1) was first derived in Camassa and Holm (1993) by using asymptotic expansions directly in the Hamiltonian for Euler's equations governing inviscid incompressible flow in the shallow water regime. It was thereby shown to be bi-Hamiltonian and integrable by the inverse scattering transform. Its periodic solutions were treated in Alber et al. (1994, 1999, 2001) and references therein. Before Camassa and Holm (1993), families of integrable equations similar to (1.1) were known to be derivable in the general context of hereditary symmetries in Fokas and Fuchssteiner (1981). However, Eq. (1.1) was not written explicitly, nor was it derived physically as a water wave equation and its solution properties were not studied before Camassa and Holm (1993). See Fuchssteiner (1996) for an insightful history of how the integrable shallow water Eq. (1.1) relates to the mathematical theory of hereditary symmetries.

Eq. (1.1) was recently re-derived as a shallow water equation by using asymptotic methods in three different approaches in Fokas and Liu (1996), in Dullin et al. (2001) and also in Johnson (2002). These three derivations used different variants of the method of asymptotic expansions for shallow water waves in the absence of surface tension. In accounting for the effects of surface tension, we shall derive an entire family of shallow water wave equations that are asymptotically equivalent at quadratic order in the shallow water expansion parameters. This is one order beyond the linear asymptotic expansion for the KdV equation. The asymptotically equivalent shallow water wave equations in this family are related amongst themselves by a continuous group of non-local transformations of variables that was first introduced by Kodama (1985a, b, 1987). We also identify four integrable soliton equations amongst the family of asymptotically equivalent shallow water equations at quadratic order.

Outline. The remainder of this section sets the context for our investigation and discusses the transformation properties of Eq. (1.1). Section 2 rederives the standard elevation field dynamics for shallow water waves following Whitham (1974). We then use an approach based on the Kodama transformation to derive Eq. (1.1) with surface tension in Section 3. Section 4 discusses the relation of Eq. (1.1) to KdV and other integrable equations, particularly KdV5, the fifth-order integrable equation in the KdV hierarchy, and another integrable non-linear equation (4.4) recently derived in Degasperis et al. (2002). Section 5 discusses the rather rich classes of travelling wave solutions for Eq. (1.1). Finally Section 6 discusses its physical relevance and the potential for measuring its special solutions.

1.1. Context

In Dullin et al. (2001) the focus was on the integrability of the equation and its isospectral properties. The derivation from Euler's equation in the case without surface tension was briefly described. Here we present the necessary details of this calculation. The present derivation also adds surface tension, which contributes to the coefficient Γ in Eq. (1.1).

In the context of water waves in the presence of surface tension there has been an increased interest in the KdV5 equation and its solitary wave solutions, see Dias and Kharif (1999) for a review. For Bond numbers $0<\sigma<1/3$ it has been shown that these solutions are not true solitary waves which decay to zero at spatial infinity. Instead, they are *generalized solitary waves* which are characterized by exponentially small ripples on their tail, as discussed, e.g., in Beale (1991). These ripples are shown in Lombardi (2000) to be exponentially small in terms of $F-1$ where $F=c/c_0$ is the Froude number. Numerical experiments of Champneys et al. (2002) suggest that in the full non-linear water wave problem there are no real solitary waves bifurcating for Bond numbers $0<\sigma<1/3$. For Bond numbers larger than 1/3, one obtains not elevations, but depressions with negative velocity.

Why derive a higher-order model equation, if these more rigorous, exact, or numerical results are already available? Or in more general terms: Which is more desirable, an exact solution of an approximate model equation, or an approximate solution of an exact equation? The family of asymptotic equations we shall derive here provides more accurate travelling waves than KdV, without requiring considerably more elaborate models. Thus, although one may obtain less information, compared to the sophisticated methods beyond all orders, the cost is also less. Including yet higher order terms in our derivation would lead to equations possessing the same exponentially small effects. Thus, one may improve the description of the shape and speed of the travelling wave without resorting to the more complicated models. That they provide only an approximation to the true solution shall be taken for granted.

Our inclusion of surface tension has a similar motivation. Although Eq. (3.16) that we shall derive has some drawbacks concerning the global properties of its dispersion relation for large k, it still gives improved descriptions for small k and small σ. Moreover, the improved simple solutions are easily obtained and analyzed.

1.2. Transformation properties

Before embarking on its derivation, we shall survey the transformation properties of Eq. (1.1). First, it is reversible, i.e., it is invariant under the discrete transformation $u(x,t)\to -u(x,-t)$. Eq. (1.1) is also Galilean *covariant*. That is, it keeps its form under transformations to an arbitrarily moving reference frame. This includes covariance under transforming to a uniformly moving Galilean frame. However, Eq. (1.1) is not Galilean *invariant*, even assuming that the momentum m Galileo-transforms in the same way as u. In fact, Eq. (1.1) transforms under

$$t\to t+t_0,\quad x\to x+x_0+ct,\quad u\to u+c+u_0,\quad m\to m+c+u_0, \tag{1.2}$$

to the form

$$m_t+um_x+2u_xm+(c_0+u_0)m_x+2u_x(c+u_0)+\Gamma u_{xxx}=0. \tag{1.3}$$

Thus, Eq. (1.1) is invariant under space and time translations (constants x_0 and t_0), covariant under Galilean transforms (constant c), and acquires linear dispersion terms under velocity shifts (constant u_0). The dispersive term u_0m_x introduced by the constant velocity shift $u_0\neq 0$ breaks the reversibility of Eq. (1.1).

Under scaling transformations of x, t and u, the coefficients of Eq. (1.1) can be changed. However, such scaling leaves the following coefficient ratios invariant,

$$C(u_xu_{xx}):C(uu_{xxx})=2:1, \tag{1.4}$$

$$C(u_{xxt})C(uu_x):C(uu_{xxx})\,C(u_t)=3:1, \tag{1.5}$$

where $C(T)$ stands for the coefficient of the term T in the scaled equation. It is pertinent to mention that the above ratios are crucial for the integrability of Eq. (1.1) Dullin et al. (2001). See also Eq. (4.4) and its discussion in Section 4.1.

2. Derivation of the η equation

Our derivation of Eq. (1.1) proceeds from the physical shallow water system along the lines of Whitham (1974). Consider water of depth $h=h_0+\eta(x,t)$ where h_0 is the mean depth, so that $z=-h_0$ at the flat bottom and $z=0$ at the free surface in equilibrium. Denote by u_{h} and u_{v} the horizontal and vertical velocity components, respectively. The z-momentum balance is

$$\frac{\mathrm{D}u_{\mathrm{v}}}{\mathrm{D}t} = -g - \frac{1}{\rho}\,\partial_z p \quad \text{with } p = \tilde{\sigma}\,\frac{h_{xx}}{(1+h_x^2)^{3/2}}, \tag{2.1}$$

where g is the constant of gravity and $\tilde{\sigma}$ is the surface tension. At the free surface the boundary condition is

$$\frac{\mathrm{D}\eta}{\mathrm{D}t} = u_{\mathrm{v}} \quad \text{at } z=\eta. \tag{2.2}$$

Introducing the potential velocity $\mathbf{u}(x,z,t)=\nabla\varphi$ we have $u_{\mathrm{h}}=\varphi_x$ and $u_{\mathrm{v}}=\varphi_z$ for the horizontal and vertical velocity components. The velocity potential φ must satisfy Laplace's equation in the interior. Eq. (2.2) gives the kinematic boundary condition for the free surface

$$\eta_t + \varphi_x\eta_x = \varphi_z \quad \text{at } z=\eta.$$

Eq. (2.1) can now be integrated to yield the dynamic boundary condition

$$\varphi_t + \frac{1}{2}(\varphi_x^2+\varphi_z^2) = -gh - \frac{1}{\rho}\,p \quad \text{at } z=\eta.$$

The equations for a fluid are written in a non-dimensionalized form by introducing $x=l_x x'$, $z=h_0 z'$, $t=(l_x/c_0)t'$, $\eta=a\eta'$ and $\varphi=(gl_x a/c_0)\varphi'$, where $c_0=\sqrt{gh_0}$. We are interested in weakly non-linear small amplitude waves in a shallow water environment and introduce the small parameters $\varepsilon=a/h_0$ and $\delta^2=(h_0/l_x)^2$ where $\varepsilon \geqslant \delta^2 > \varepsilon^2 \geqslant \varepsilon\delta^2 \geqslant \delta^4$. Upon omitting the primes and expanding the pressure term up to order $\varepsilon^2\delta^2$, the Euler equations and the boundary conditions at the free surface and at the bottom take the form

$$\delta^2\varphi_{xx} + \varphi_{zz} = 0 \quad \text{in } -1<z<\varepsilon\eta, \tag{2.3}$$

$$\eta_t + \varepsilon\varphi_x\eta_x - \frac{1}{\delta^2}\,\varphi_z = 0 \quad \text{at } z=\varepsilon\eta, \tag{2.4}$$

$$\eta + \varphi_t + \frac{1}{2}\left(\varepsilon\varphi_x^2 + \frac{\varepsilon}{\delta^2}\,\varphi_z^2\right) - \sigma\delta^2\eta_{xx} \quad \text{at } z=\varepsilon\eta, \tag{2.5}$$

$$\varphi_z = 0 \quad \text{at } z=-1, \tag{2.6}$$

where $\sigma=\tilde{\sigma}/(h_0\rho c_0^2)$ is the dimensionless Bond number. The ordering of ε and δ^2 is as specified, provided that $\sigma=\mathcal{O}(1)$.

The equation for the interior is identically satisfied by the ansatz,

$$\varphi(x,z,t) = \sum_{m=0}^{\infty} \delta^{2m}(-1)^m \frac{z^{2m}}{(2m)!} \frac{\partial^{2m} f(x)}{\partial x^{2m}}, \tag{2.7}$$

with an arbitrary function $f(x)$. The velocity potential φ is expanded at the mean height $z = z_0$, where $\psi = \varphi(x, z_0)$, up to order $\mathcal{O}(\delta^6)$ as

$$\varphi = \psi - \tfrac{1}{2}\,\delta^2(z - z_0)(z + z_0)\psi_{xx} + \tfrac{1}{24}\,\delta^4(z - z_0)(z + z_0)(z^2 - 5z_0^2)\psi_{xxxx}$$
$$- \tfrac{1}{720}\,\delta^6(z - z_0)(z + z_0)(z^4 - 14zz_0 + 61z_0^4)\psi_{xxxxxx}. \tag{2.8}$$

For an expansion at $z = z_0 = 0$, in fact $f = \psi$. We will present the derivation with this choice, in order to make it as transparent as possible. However, the derivation would yield the same result when expanding at an arbitrary height $z = z_0$ (see the comments at the end of this section). We have included one more order in δ^2 in this expansion because in (2.4) ϕ_z is divided by δ^2. The dynamics of the free surface is entirely determined by the upper boundary conditions which to order $\mathcal{O}(\delta^4)$ read

$$\eta_t + \psi_{xx} + \varepsilon(\eta_x\psi_x + \eta\psi_{xx}) - \frac{\delta^2}{6}\psi_{xxxx} - \frac{\varepsilon\delta^2}{2}(\eta_x\psi_{xxx} + \eta\psi_{xxxx}) - \frac{\delta^4}{120}\psi_{xxxxxx} = 0$$

$$\eta + \psi_t + \frac{\varepsilon}{2}\psi_x^2 - \frac{\delta^2}{2}(\psi_{xxt} + 2\sigma\eta_{xx}) - \frac{\varepsilon\delta^2}{2}(2\eta\psi_{xxt} + \psi_x\psi_{xxx} - \psi_{xx}^2) + \frac{\delta^4}{24}\psi_{xxxxxt} = 0.$$

Differentiating the second equation with respect to x and introducing the horizontal velocity $w = \psi_x$ at the mean height $z = 0$ yields the following set of equations in conservation form

$$\eta_t + w_x + \varepsilon(w\eta)_x - \frac{\delta^2}{6}w_{xxx} - \frac{\varepsilon\delta^2}{2}(w_{xx}\eta)_x - \frac{\delta^4}{120}w_{xxxxx} = 0,$$
$$w_t + \eta_x + \frac{\varepsilon}{2}(w^2)_x - \frac{\delta^2}{2}(w_{xt} + 2\sigma\eta_{xx})_x - \frac{\varepsilon\delta^2}{2}(2\eta w_{xt} - w_x^2 + ww_{xx})_x + \frac{\delta^4}{24}w_{xxxxt} = 0. \tag{2.9}$$

Note that the terms to order $\mathcal{O}(\varepsilon)$ are the well known shallow water equations.

As in the derivation of the KdV equation in Whitham (1974) we now restrict to unidirectional waves by assuming a relationship $w = \eta + \epsilon f[\eta]$ between w, the horizontal velocity at the mean height, and the elevation, η. The functional f shall be determined so that the two equations in (2.9) both reduce to the same single equation for the height field η. The relation $w = \eta + \epsilon f[\eta]$ can be considered as an approximate reduced manifold which is tangent to the space of linear waves moving to the right. Note: allowing leftward travelling waves would violate the hypothesis that such a relation exists at order ε^4, due to a coupling between the left- and right-going waves. See Prasad and Akylas (1997); Marchant (2002); Schneider and Wayne (2000) for discussions of this point. We expand w in a power-series of ε, δ and include terms of order $\mathcal{O}(\varepsilon\delta^2)$ and $\mathcal{O}(\delta^4)$ to find

$$w = \eta + \varepsilon A(\eta) + \delta^2 B(\eta) + \varepsilon^2 C(\eta) + \varepsilon\delta^2 D(\eta) + \delta^4 E(\eta). \tag{2.10}$$

Upon ordering in powers of the small parameters the coefficients in the transformation (2.10) are determined by requiring that both boundary conditions in (2.9) are satisfied simultaneously. We find

$$\mathcal{O}(\varepsilon): A_x + \eta\eta_x = A_t, \tag{2.11}$$

$$\mathcal{O}(\delta^2):\ \partial_x(B - \tfrac{1}{6}(1-6\sigma)\eta_{xx}) = \partial_t(B - \tfrac{1}{2}\eta_{xx}), \tag{2.12}$$

$$\mathcal{O}(\varepsilon^2):\ C_x = C_t, \tag{2.13}$$

$$\mathcal{O}(\varepsilon\delta^2):\ \partial_x(D - \tfrac{1}{6}A_{xx} - \tfrac{1}{2}\eta_x^2) = \partial_t(D - \tfrac{1}{2}A_{xx} - \tfrac{1}{2}\eta_x^2) - \eta\eta_{xxt}, \tag{2.14}$$

$$\mathcal{O}(\delta^4):\ \partial_x(E - \tfrac{1}{6}B_{xx} + \tfrac{1}{120}\eta_{xxxx}) = \partial_t(E - \tfrac{1}{2}B_{xx} + \tfrac{1}{24}\eta_{xxxx}). \tag{2.15}$$

The first two Eqs. (2.11) and (2.12) are readily solved by

$$A = -\tfrac{1}{4}\eta^2 \quad \text{and} \quad B = \tfrac{1}{6}(2-3\sigma)\eta_{xx}. \tag{2.16}$$

At the next iteration of the expansion, the time-derivatives

$$\begin{aligned}\eta_t &= -w_x - \varepsilon(\eta_x w + \eta w_x) + \frac{\delta^2}{6} w_{xxx} \\ &= -\eta_x - \tfrac{3}{2}\varepsilon\eta\eta_x - \tfrac{1}{6}\delta^2(1-3\sigma)\eta_{xxx},\end{aligned} \tag{2.17}$$

that appear in Eqs. (2.11) and (2.12) generate higher order terms which modify the higher order balances (2.13)–(2.15). These then lead to

$$C = \tfrac{1}{8}\eta^3, \quad D = \tfrac{1}{16}(3+7\sigma)\eta_x^2 + \tfrac{1}{4}\eta\eta_{xx}, \quad E = \tfrac{1}{120}(12 - 20\sigma - 15\sigma^2)\eta_{xxxx}. \tag{2.18}$$

Under the transformation (2.10) both equations in (2.9) are equivalent to the desired order and one finds a single decoupled equation for the height field, or elevation, η,

$$\begin{aligned}&\eta_t + \eta_x + \tfrac{3}{2}\varepsilon\eta\eta_x + \tfrac{1}{6}\delta^2(1-3\sigma)\eta_{xxx} - \tfrac{3}{8}\varepsilon^2\eta^2\eta_x + \varepsilon\delta^2(\tfrac{1}{24}(23+15\sigma)\eta_x\eta_{xx} \\ &\quad + \tfrac{1}{12}(5-3\sigma)\eta\eta_{xxx}) + \delta^4\tfrac{1}{360}(19 - 30\sigma - 45\sigma^2)\eta_{xxxxx} = 0.\end{aligned} \tag{2.19}$$

This is a well known result, which up to this order has been derived for example by Marchant and Smyth (1990) and recently in Johnson (2002) without surface tension. Note that the same elevation equation would be obtained by expanding the potential not about the mean depth $z_0 = 0$, but about an arbitrary depth z_0, see Olver (1983) and Kirby (1997). This still holds, even when surface tension is included. Thus, the height field equation in this approximation is independent of the depth at which the horizontal velocity is measured. For simplicity, we have chosen to evaluate the velocity potential φ at $z_0 = 0$. As a matter of fact, the same elevation equation would also hold if we had chosen a vertically averaged potential to determine the velocity field u, as done in Wu and Ted (1998).

From the point of view of physically meaningful interpretations of solutions of equation (2.19), z_0 is the depth at which velocity measurements are taken. The above discussion shows that the evolution equation of the height field (2.19) is independent of this arbitrary measurement location z_0. However, the transformation (2.10) does explicitly contain z_0 in relating the height field η to the measured velocity w.

3. Transformation to an integrable equation

Eq. (1.1) will emerge as being asymptotically equivalent to Eq. (2.19) after two further steps. First, we shall perform a near-identity transformation,

$$\eta = \eta[u] = u + \varepsilon f[u] + \delta^2 g[u],$$

relating the wave elevation and a velocity-like quantity, u. One may consider u as an auxiliary quantity in which the transformed equation becomes particularly simple. To obtain the physically meaningful quantity, η, one must transform back, see below. The functionals f and g are to be chosen so that they generate the terms proportional to $uu_x, u_xu_{xx}, uu_{xxx}$ and u_{xxx} in Eq. (1.1), afterwards we apply the Helmholtz operator $H = 1 - v\delta^2\partial_x^2$ which generates the u_{xxt} term. As in Kodama (1985a, b, 1987) the functional $g[u]$ is proportional to u_{xx} and $f[u]$ is a linear combination of u^2 and the non-local term $u_x\partial^{-1}$, where ∂^{-1} means integration in x. Thus, together with the parameter v there are four coefficients in this transformation. These shall be chosen so that Eq. (1.1) emerges after a rescaling of u, x and t.

The near-identity Kodama transformation depends on three parameters α_1, α_2 and β,

$$\eta = \eta[u] = u + \varepsilon(\alpha_1 u^2 + \alpha_2 u_x \partial^{-1} u) + \delta^2 \beta u_{xx}. \tag{3.1}$$

Terms of degree n in the expansion parameters ε and δ^2 start contributing at degree $n+1$ in the transformed equation. Therefore no terms of quadratic order are needed in the transformation. Inserting transformation (3.1) into Eq. (2.19) for the height field η leads at each order to

$$\begin{aligned}
&\mathcal{O}(1)\colon\ u_t + u_x,\\
&\mathcal{O}(\varepsilon)\colon\ 2\alpha_1 uu_t + 2\alpha_1 uu_x + \alpha_2(u_{xt}\partial^{-1}u + u_{xx}\partial^{-1}u + u_x\partial^{-1}u_t + uu_x) + \tfrac{3}{2}\,uu_x,\\
&\mathcal{O}(\delta^2)\colon\ \beta u_{xxt} + u_{xxx}(\beta + \tfrac{1}{6} - \tfrac{1}{2}\,\sigma),\\
&\mathcal{O}(\varepsilon^2)\colon\ \tfrac{9}{2}\,\alpha_1 u^2 u_x + \tfrac{3}{2}\,\alpha_2(u^2u_x + uu_{xx}\partial^{-1}u + u_x^2\partial^{-1}u) - \tfrac{3}{8}\,u^2u_x,\\
&\mathcal{O}(\varepsilon\delta^2)\colon\ (\tfrac{23}{24} + \tfrac{5}{8}\sigma + \tfrac{1}{3}(3\alpha_1 + 2\alpha_2)(1-3\sigma) + \tfrac{3}{2}\,\beta)u_xu_{xx}\\
&\qquad + (\tfrac{5}{12} - \tfrac{1}{4}\,\sigma + \tfrac{1}{6}(2\alpha_1 + 3\alpha_2)(1-3\sigma) + \tfrac{3}{2}\,\beta)uu_{xxx} + \tfrac{1}{6}\,\alpha_2(1-3\sigma)u_{xxxx}\partial^{-1}u,\\
&\mathcal{O}(\delta^4)\colon\ (\tfrac{1}{6}\,\beta_2(1-3\sigma) + \tfrac{1}{360}(19 - 30\sigma - 45\sigma^2))u_{xxxxx}.
\end{aligned} \tag{3.2}$$

As before, we expand the time derivatives to linear order as

$$u_t = -u_x - \tfrac{3}{2}\,\varepsilon uu_x - \tfrac{1}{6}\,\delta^2(1-3\sigma)u_{xxx}, \tag{3.3}$$

$$\begin{aligned}
u_{xt} &= -u_{xx} - \tfrac{3}{2}\,\varepsilon u_x^2 - \tfrac{3}{2}\,\varepsilon uu_{xx} - \tfrac{1}{6}\,\delta^2(1-3\sigma)u_{xxxx},\\
u_{xxt} &= -u_{xxx} - \tfrac{9}{2}\,\varepsilon u_xu_{xx} - \tfrac{3}{2}\,\varepsilon uu_{xxx}.
\end{aligned} \tag{3.4}$$

This expansion generates higher order terms, leading to

$\mathcal{O}(1)$: $u_t + u_x$,

$\mathcal{O}(\varepsilon)$: $\frac{3}{2} uu_x$,

$\mathcal{O}(\delta^2)$: $\frac{1}{6}(1-3\sigma)u_{xxx}$,

$$\mathcal{O}(\varepsilon^2)\text{: } (\tfrac{3}{2}\alpha_1 + \tfrac{3}{4}\alpha_2 - \tfrac{3}{8})u^2u_x, \tag{3.5}$$

$$\mathcal{O}(\varepsilon\delta^2)\text{: } \tilde{A}u_xu_{xx} + \tilde{B}uu_{xxx}, \tag{3.6}$$

$$\mathcal{O}(\delta^4)\text{: } \tfrac{1}{360}(19 - 30\sigma - 45\sigma^2)u_{xxxxx}, \tag{3.7}$$

where in (3.6) we used

$$\tilde{A} = \tfrac{23}{24} + \tfrac{5}{8}\sigma + \tfrac{1}{2}(2\alpha_1 + \alpha_2)(1-3\sigma) - 3\beta \quad \text{and} \quad \tilde{B} = \tfrac{5}{12} - \tfrac{1}{4}\sigma + \tfrac{1}{2}\alpha_2(1-3\sigma).$$

The first step of the derivation is now complete. In the second step, applying the Helmholtz operator $H = 1 - v\delta^2\partial_x^2$ creates terms with two more x derivatives multiplied by δ^2. In particular the terms of order $\mathcal{O}(\varepsilon^2)$ are unchanged. These terms are proportional to u^2u_x and they must vanish for Eq. (1.1) to emerge. The application of the Helmholtz operator simply recreates the u_{xxt} term that had previously been eliminated. Alternatively, the same Eq. (1.1) could be obtained by splitting the time derivative, that is, by partially substituting the time derivative u_{xxt} in (3.2) using its asymptotic approximations (3.4), as in Peregrine (1966). The order $\mathcal{O}(\varepsilon^2)$ coefficient will vanish, provided the parameters α_1 and α_2 are chosen to satisfy

$$4\alpha_1 + 2\alpha_2 = 1. \tag{3.8}$$

The order $\mathcal{O}(\delta^4)$ terms receive a contribution that arises from applying the Helmholtz operator to the terms of order $\mathcal{O}(\delta^2)$, and this combination has to vanish, so that the final equation does not possess a u_{xxxxx} term. This requirement determines v as

$$v = \frac{1}{60}\,\frac{19 - 30\sigma - 45\sigma^2}{1 - 3\sigma} \tag{3.9}$$

and in the following we shall consider v to be given by this function of σ. Note that removal of the highest order term was made possible by introducing the additional parameter v via the Helmholtz operator. The remaining terms containing free parameters α_2 and β are of order $\varepsilon\delta^2$ and they combine additively as

$$(\tilde{A} - \tfrac{9}{2}v)u_xu_{xx} + (\tilde{B} - \tfrac{3}{2}v)uu_{xxx}.$$

To ensure equivalence to (1.1) except for scaling we need the relative coefficients to appear in the ratio (1.4), so that

$$(\tilde{A} - \tfrac{9}{2}v) : (\tilde{B} - \tfrac{3}{2}v) = 2:1. \tag{3.10}$$

In addition we also need to satisfy (1.5), so that

$$\tfrac{3}{2}v : (\tilde{B} - \tfrac{3}{2}v) = 3:1.$$

These two conditions imply $\tilde{B}=2\nu$ and $\tilde{A}=11\nu/2$. As a result we finally obtain the equation

$$u_t - \nu\delta^2 u_{xxt} + u_x + \tfrac{3}{2}\,\varepsilon u u_x - \tfrac{1}{2}\,\varepsilon\delta^2\nu(u u_{xxx} + 2u_x u_{xx}) + \delta^2(\tfrac{1}{6} - \nu - \tfrac{1}{2}\,\sigma)u_{xxx} = 0, \tag{3.11}$$

which can be rewritten in terms of $m = u - \nu\delta^2 u_{xx}$ as

$$m_t + m_x + \frac{\varepsilon}{2}(u m_x + 2m u_x) + \delta^2\left(\frac{1}{6} - \frac{1}{2}\,\sigma\right) u_{xxx} = 0. \tag{3.12}$$

The explicit coefficients in the Kodama transformation (3.1) are thus

$$\alpha_1 = \frac{7}{20} - \sigma\,\frac{1}{5}\,\frac{2-3\sigma}{(1-3\sigma)^2}, \tag{3.13}$$

$$\alpha_2 = -\frac{1}{5} + \sigma\,\frac{2}{5}\,\frac{2-3\sigma}{(1-3\sigma)^2}, \tag{3.14}$$

$$\beta = \frac{1}{30} - \sigma\,\frac{1}{30}\,\frac{17-30\sigma}{1-3\sigma}. \tag{3.15}$$

Scaling back to physical variables where u has units of φ_x which are $ga/c_0 = c_0 a/h_0$ gives

$$m_t + c_0 m_x + \tfrac{1}{2}(u m_x + 2m u_x) + \Gamma u_{xxx} = 0, \tag{3.16}$$

where $m = u - \nu h^2 u_{xx}$ and $\Gamma = c_0 h^2(1-3\sigma)/6$. By an additional scaling of u by 2 this can be reduced to the canonical form (1.1). The parameters α^2 and Γ in (1.1) are given in terms of physical variables as

$$\alpha^2 = \nu h^2 = h^2\,\frac{1}{60}\,\frac{19 - 30\sigma - 45\sigma^2}{1-3\sigma}, \tag{3.17}$$

$$\Gamma = \frac{c_0 h^2}{6}(1-3\sigma). \tag{3.18}$$

The parameter Γ changes sign when the Bond number σ crosses the critical value 1/3. For later reference we record the values of $\sigma > 0$ for which α^2 or $\Gamma - c_0\alpha^2$ vanish as

$$\sigma_\alpha = -\tfrac{1}{3} + \tfrac{2}{15}\sqrt{30} \approx 0.39696, \quad \sigma_\gamma = \tfrac{1}{9} + \tfrac{4}{45}\sqrt{10} \approx 0.39220, \tag{3.19}$$

respectively.

In the special case $c_0 = \Gamma = 0$, Eq. (3.16) is called the "peakon equation." Its *peakon* solutions are solitary waves whose derivative is discontinuous at the extremum. These solution were introduced and discussed in Camassa and Holm (1993). However, the peakon equation is a zero-dispersion case that does not strictly follow as a water wave equation in a weakly non-linear shallow approximation from the Euler equation by this technique. Neither a Galilean transformation nor an appropriate splitting can eliminate the two linear dispersive terms simultaneously. One is always left with a residual linear dispersion, whose final removal requires the additional velocity shift, u_0, appearing in transformation (1.2).

Johnson (2002) has recently derived the CH equation as a shallow water wave equation in a superficially similar way. However, there are fundamental differences between the derivations here and in Johnson (2002). First, the derivation in Johnson (2002) involves the evaluation of the potential

(2.8) at a particular height $z_0 = 1/\sqrt{2}$. Instead of the height z_0, the free parameters in the Kodama transformation (3.1) are used here to obtain the desired equation. But a deeper issue is involved in distinguishing between the two different derivations. In Johnson (2002) the fifth-order derivative, which is an essential part of our derivation, was omitted. In Section 4 we shall show that the CH equation is asymptotically equivalent to the KdV5 equation, which involves the fifth-order derivative. In obtaining (3.16), the free parameters in the Kodama transformation (3.1) and in the Helmholtz operator were used in transforming away the fifth-order derivative. However, in Johnson (2002) this term was simply omitted by using a scaling of ε and δ that does not allow for $\varepsilon = \delta^2$. However, the particular scaling $\varepsilon = \delta^2$ cannot be discarded, as it assures the primary balance of linear dispersion and non-linear steepening in the KdV equation and is, hence, the backbone for the lower order balance of the higher order equation (3.16). The scaling we employ and the transformations we use yield the same result and, moreover, also allow one to study the KdV5 equation (see Section 4).

In order to compare predictions and to compare solutions to physically measurable quantities, the solutions for the horizontal velocity-like variable u must be transformed back to the elevation field η by using (3.1). However, the derivation not only used transformation (3.1) but it also involved application of the Helmholtz operator. Therefore one should check that it is sufficient to simply invert (3.1). Fortunately, when the inverse transformation $u = u[\eta]$ of the same form as (3.1) with u and η interchanged is substituted into (3.12), we find that the coefficients just reverse their signs. We conclude that (1.1) is equivalent to the shallow water wave equation (2.19) up to and including terms of order $\mathcal{O}(\delta^4)$.

4. Relation to other integrable equations

The Kodama transformation can also be used to transform the CH equation into the integrable fifth order KdV equation (henceforth called KdV5). Li and Sibgatullin (1997) show that Eq. (2.19) for the elevation η can be transformed into the KdV5 equation. Here we shall show that the CH equation and the KdV5 equation are also asymptotically equivalent under a Kodama transformation. To this end, we first expand the time derivative in the u_{xxt}-term using the equation itself and then apply a transformation of the form introduced in Kodama (1985a, b, 1987), namely

$$u = v + \varepsilon(\alpha_1 v^2 + \alpha_2 v_x \partial^{-1} v) + \delta^2 \beta v_{xx}. \tag{4.1}$$

Choosing the values in this Kodama transformation as

$$\alpha_1 = \frac{\alpha^2}{\Gamma}, \quad \alpha_2 = 2\frac{\alpha^2}{\Gamma}, \quad \beta = 2\alpha^2 \tag{4.2}$$

transforms the CH Eq. (1.1) into the integrable KdV5 equation

$$v_t + c_0 v_x + 3vv_x + 5(vv_{xxx} + 2v_x v_{xx})\alpha^2 + \frac{15}{2}\frac{\alpha^2 v^2 v_x}{\Gamma} + \Gamma(\alpha^2 v_{xxxxx} + v_{xxx}). \tag{4.3}$$

Transformation (4.1) is singular in the limit $\Gamma \to 0$, so that the peakon solutions of CH in this limit cannot be mapped to solutions of KdV5.

A Kodama transformation of form (4.1) cannot transform the CH equation to the KdV equation itself. However, Fokas and Liu (1996) show that such a transformation is possible, provided another term of the form xv_t is included in the Kodama transformation. Unfortunately, the term xv_t is not uniformly bounded, so we shall decline to use it. Were one to use (4.1) to transform solutions of the CH equation into solutions of the KdV5 equation, the unboundedness of this term would present a real problem when transforming travelling wave solutions, which move asymptotically in time toward $x = \pm\infty$. Moreover, as we shall see, the term xv_t would change the dispersion relation, so again its use would be problematic. In contrast, to see that transformation (4.1) does not change the dispersion relation, one may observe that only terms linear in u or its derivatives produce linear terms in the transformed equation. Similarly, non-linear terms in the equation being transformed will only create non-linear terms in the resulting transformed equation. Therefore, we may restrict to a transformation $u = v + \epsilon L(v)$ in which L is a linear differential operator with constant coefficients and the linear equation to be transformed is $u_t = M(u)$. To first order, we then have $v_t = M(v)$ and the full transformation gives

$$v_t + \epsilon L(v_t) = M(v + \epsilon L(v)).$$

Now the first-order equation may be used to eliminate the time derivatives that are not of order zero, thereby yielding

$$v_t + \epsilon L(M(v)) = M(v) + \epsilon M(L(v)).$$

If M and L commute, as they do when they have constant coefficients, the final answer is $v_t = M(v)$ so that a linear equation is unchanged. However, including a term of the form xu_t in the transformation in general will cause the operators to no longer commute and, thus, the linear equation will be changed. A proper near-identity transformation should leave the linear part of the equation invariant and only transform higher order terms. Therefore we do not include terms of the form xv_t in the transformation.

Transforming from KdV5 to CH also involves the application of the Helmholtz operator $H = 1 - v\delta^2\partial_x^2$. As we have just seen, the Kodama transformation leaves the linear part of the equation unchanged. Applying the Helmholtz operator to an equation does change the linear part, but it still leaves the dispersion relation unchanged. To see this, let the linear part of the equation be given by $u_t = M(u)$. The new equation is $H(u_t) = H(M(u))$. If H and M are linear with constant coefficients this gives $H(u)_t = M(H(u))$ so that with the definition $m = H(u)$ we obtain $m_t = M(m)$, which has the same dispersion relation. Note that this is not true if we truncate higher order terms in $H(M(u))$. If we truncate, then the dispersion relation will agree up to the order of truncation. For example, the dispersion relation for (1.1) is a rational function, which differs from the polynomial dispersion relation obtained from (2.19). However, by the above argument the two agree up to the desired order.

We conclude that (1.1) is asymptotically equivalent to the integrable KdV5 equation, and both of them are equivalent to (2.19) at order $\mathcal{O}(\delta^4)$. However, the equivalence of (1.1) to the KdV5 equation breaks down in the limit $\Gamma \to 0$, because the transformation as well as the resulting equation contains terms divided by Γ. Therefore, the peakon equation cannot be transformed into KdV5.

Using the additional parameter supplied by the Helmholtz operator allows for the removal of the highest order term while preserving the dispersion relation, which is unchanged by applying a linear operator to the equation. One advantage of the CH equation over the asymptotically equivalent KdV5 equation is that it is easier to integrate numerically because it does not contain the fifth derivative. This is in accordance with the general smoothing effect of the Helmholtz operator.

4.1. The b-equation

Recently a new variant of (1.1) was introduced in Degasperis et al. (2002) as

$$m_t + um_x + bu_x m = c_0 u_x - \Gamma u_{xxx}, \tag{4.4}$$

where b is an arbitrary parameter. The solutions of the b-equation (4.4) were studied numerically for various values of b in Holm and Staley (2003a, b), where the parameter b was taken as a bifurcation parameter. The cases $b=2$ and 3 are special values for the b-equation (4.4). The case $b=2$ restricts the b-equation to the integrable CH equation of Camassa and Holm (1993). The case $b=3$ in (4.4) recovers the DP equation of Degasperis and Procesi (1999), which was shown to be integrable in Degasperis et al. (2002). These two cases exhaust the integrable candidates for (4.4), as was shown using Painlevé analysis in Degasperis et al. (2002). The b-family of equations (4.4) was also shown in Mikhailov and Novikov (2002) to admit the symmetry conditions necessary for integrability only in the cases $b=2$ for CH and $b=3$ for DP.

We shall show here that the b-equation (4.4) can also be obtained from the shallow water elevation equation (2.19) by an appropriate Kodama transformation. The derivation in the previous section is essentially unchanged up to Eq. (3.10). The two scaling relations (1.4, 1.5) now read

$$(\tilde{A} - \tfrac{9}{2}v) : (\tilde{B} - \tfrac{3}{2}v) = b : 1,$$

$$\tfrac{3}{2}v : (\tilde{B} - \tfrac{3}{2}v) = b + 1 : 1.$$

These two conditions imply

$$\tilde{B} = v\,\frac{3}{2}\,\frac{b+2}{b+1} \quad \text{and} \quad \tilde{A} = v\,\frac{3}{2}\,\frac{4b+3}{b+1}.$$

The resulting Kodama transformation of the form (3.1) with coefficients α_1', α_2', and β' are

$$\alpha_1' = \alpha_1 + 3\Lambda$$

$$\alpha_2' = \alpha_2 - 6\Lambda$$

$$\beta' = \beta - (1 - 3\sigma)\Lambda$$

where

$$\Lambda = \frac{b-2}{b+1}\,\frac{45\sigma^2 + 30\sigma - 19}{360}.$$

Therefore any $b \neq -1$ may be achieved by an appropriate Kodama transformation. Note that when $\sigma = \sigma_\alpha$, see (3.19), then $\alpha^2 = 0$, hence $\Lambda = 0$ is independent of b. After this transformation (4.4) is obtained by further scaling of the new dependent variable u by the factor $b+1$. See Holm and Staley (2003a, b) for discussions of Eq. (4.4) in which b is treated as a bifurcation parameter when $c_0 = 0$ and $\Gamma = 0$.

We conclude that the detailed values of the coefficients of the asymptotic analysis hold only modulo the Kodama transformations and these transformations may be used to advance the analysis and thereby gain insight. Thus, the Kodama-transformations may provide an answer to the perennial question “Why are integrable equations so ubiquitous when one uses asymptotics in modelling?”

5. Travelling wave solutions

The water wave equation (3.16) can be viewed as a hybrid of two different integrable limiting equations. On the one hand, the limit $\alpha^2=0$ leads to the KdV equation when including terms of order ε and δ^2 which supports regular solitons. Thus, the primary physical mechanism for the propagation of solitary shallow water waves at order $\mathcal{O}(\varepsilon, \delta^2)$ is the balance between non-linear steepening and linear dispersion. However, the CH equation (3.16) introduces additional higher order combinations of balance, including the non-linear/non-local balance in the following (rescaled) zero-dispersion case, whose non-linear dynamics still remains, even in the limit of vanishing linear dispersion, i.e. $c_0 = 0 = \Gamma$, namely,

$$m_t + um_x + 2mu_x = 0, \quad \text{with } m = u - \alpha^2 u_{xx}. \tag{5.1}$$

This non-linear/non-local balance produces a confined solitary travelling wave pulse,

$$u(x,t) = c\mathrm{e}^{-|x-ct|/\alpha},$$

called the *peakon*. In the momentum variable m it is given by a δ-function at $x - ct$. The peakon travels with speed equal to its peak amplitude. This solution is non-analytic, having a jump in derivative at its peak. Peakons are true solitons that interact via elastic collisions under Eq. (5.1), as discussed in Camassa and Holm (1993). Whereas the KdV equation has purely linear dispersion, its extension the Camassa–Holm (CH) equation (3.16) possesses the peakon limit (5.1) which evolves by non-linear balance. Other non-classical solutions such as the travelling waves of compact support called compactons in Li et al. (1999) also exist in this equation and we will discuss their parameter dependence in the following.

5.1. Dispersion relation

The interplay between the local and non-local linear dispersion in the CH equation (1.1), or (3.16), is evident in its phase velocity relation,

$$\frac{\omega}{k} = c_0 - \frac{\Gamma k^2}{1 + \alpha^2 k^2}, \tag{5.2}$$

for waves with frequency ω and wave number k linearized around $u = 0$. For $\Gamma < 0$, short waves and long waves travel in the same direction. Long waves travel faster than short ones (as required in shallow water) provided $\Gamma < 0$. Then the phase velocity lies in $\omega/k \in (c_0 - \Gamma/\alpha^2, c_0]$. At low wave numbers, the constant dispersion parameters α^2 and Γ perform rather similar functions. At high wave numbers, however, the parameter α^2 keeps the phase velocity of the wave properly bounded and the dispersion relation is similar to the original dispersion relation for water waves, provided that the surface tension vanishes and $\sigma = 0$ (see Section 6). Its remarkably accurate linear dispersion properties give the CH equation (1.1) a clear advantage over the KdV equation (provided $\sigma = 0$). We note that radiation is absent in the peakon equation—in this case, linear dispersion is absent ($c_0=\Gamma=0$). For non-vanishing surface tension the dispersion relation describing shallow water waves is unbounded for large wave numbers, whereas the dispersion relation of Eq. (1.1) saturates to the asymptotic value $c_0 - \Gamma/\alpha^2$. This property makes (1.1) inferior to KdV5 for non-zero surface tension. The linear dispersion relation of KdV5 and its unboundedness for high wave numbers allows for

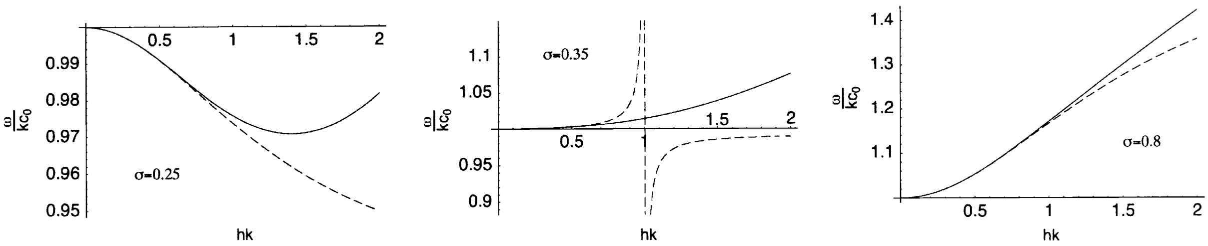

Fig. 1. Normalized dispersion relations for $\sigma = 0.25, 0.35, 0.8$ as a function of hk. The full line is the exact phase speed (5.3); the dashed line is the approximation (5.2).

resonances of a supercritical solitary wave with high wave number linear waves which give rise to exponentially small ripples at the tails of the solitary wave in accordance with the study of water wave solution of the full Euler equation, see for example Beale (1991), Grimshaw and Joshi (1995), Dias and Kharif (1999) and Lombardi (2000). Nevertheless, when k and σ are small, we do obtain improved results regarding the shape and speed of travelling waves, as discussed below (Fig. 1).

The connection to the physical parameters $\alpha = \alpha(\sigma)$ and $\Gamma = \Gamma(\sigma)$ is defined in Eqs. (3.17) and (3.18). Eq. (3.16) was derived from the original water wave problem by means of two transformations (2.10) and (3.1), respectively, and the application of the Helmholtz operator. Only the linear terms of (2.10) and (3.1) and the linear Helmholtz operator could alter the dispersion relation, but since the transformations and the Helmholtz operator are applied to the whole equation, the original linear dispersion relation is not altered at all, independently of the actual choice of the coefficients of the transformation and of the parameter v, see the discussion in Section 4. Therefore the dispersion relation (5.2) matches the dispersion relation for water waves up to quintic order. For comparison, the dispersion relation for water waves developed for small wave number k is

$$\frac{\omega}{c_0 k} = \sqrt{\frac{1+\sigma h^2k^2}{hk}\tanh hk} \tag{5.3}$$

$$\approx 1 - \tfrac{1}{6}(1-3\sigma)h^2k^2 + \tfrac{1}{360}(19 - 30\sigma - 45\sigma^2)h^4k^4 \tag{5.4}$$

$$\approx 1 - \tfrac{1}{6}(1-3\sigma)h^2k^2(1 - vh^2k^2). \tag{5.5}$$

Therefore the dispersion relations are in agreement up to fifth order in wave steepness hk.

5.2. Reduction to an ODE

Travelling wave solutions are obtained by the ansatz $u(x,t) = u(s)$, with $s = x - ct$. We obtain after integration,

$$u''(\Gamma + \alpha^2(\Delta - u)) = \Delta u - \frac{3}{2}u^2 + \frac{\alpha^2}{2}u'^2, \tag{5.6}$$

where the prime denotes differentiation with respect to s. We note that the dispersion coefficient Γ is the dispersion coefficient in the equations after transformation into the rest frame, and Δ is the difference in the wave speeds $\Delta = c - c_0$. This system follows from the canonical equations with Hamiltonian function

$$\mathscr{H}(u,p) = \frac{p^2}{2(\Gamma + \alpha^2(\Delta - u))} - \frac{\Delta}{2}u^2 + \frac{1}{2}u^3, \tag{5.7}$$

where the momentum p canonically conjugate to u is defined by $p = (\Gamma + \alpha^2(\Delta - u))u'$. Regular equilibria are defined by $\partial\mathscr{H}/\partial u = \partial\mathscr{H}/\partial p = 0$. They have $p = 0$ and $u = 0$ or $u = 2\Delta/3$. There also can exist singular equilibria defined by the vanishing of $\Gamma + \alpha^2(\Delta - u)$ and in addition the right-hand side of (5.6).

To include the singular equilibria in the discussion it is better to use the energy expressed in terms of the variables u and u' instead of the canonical variables u and p in the Hamiltonian. In addition we perform a scaling of u and s such that

$$u = \Delta\tilde{u}, \quad u' = \frac{\Delta}{|\alpha|}\tilde{u}'. \tag{5.8}$$

After this scaling the single remaining dimensionless parameter is

$$r = \frac{\Gamma}{|\alpha|^2\Delta}. \tag{5.9}$$

Note that if $\Delta < 0$ then this rescaling changes elevations travelling to the left into depressions travelling to the right, and vice versa. This transformation is not canonical; however, for discussing critical points this is not important. Scaling the energy with Δ^3 yields (after dropping the tildes)

$$E_\pm(u,u') = \tfrac{1}{2}((r \pm (1-u))u'^2 - u^2 + u^3). \tag{5.10}$$

Here the subscript $\pm$ denotes the sign of α^2. The equilibria are now given by the critical points of the energy (5.10). We note that $\alpha^2 < 0$ is admitted; in our derivation of Eq. (3.16) we defined $\alpha^2 = vh^2$ (3.17). Thus the sign of α^2 is entirely determined by v which is defined in (3.9), so that $v < 0$ for $\sigma \in (1/3, \sigma_\alpha)$, see (3.19). We note that for negative v the Helmholtz operator is not smoothing anymore. Also note that the dispersion relation possesses a pole in this case. Non-invertibility of the Helmholtz operator in the case $\alpha^2 < 0$ does not exclude this case. In the transformations used for the derivation of Eq. (3.16), the inverse Helmholtz operator does not occur. Eq. (3.16) can also be transformed into Eq. (2.19) (or into (4.3)) by means of re-substituting time derivatives.

The parameter r allows a classification of different solution types in a simple way. $E_\pm$ has two critical points which are independent of r, namely

$$u' = 0 \quad \text{and either} \quad u = 0, \text{ or } u = \tfrac{2}{3}, \tag{5.11}$$

with corresponding critical values of the energy 0 and $-2/27$, respectively. In general, the solution types are different for positive and negative α^2.

5.3. Case 1: $\alpha^2 > 0$

This case possesses two r-dependent singular critical points for which

$$u' = \pm\sqrt{3u_c^2 - 2u_c} \qquad \text{and} \qquad u = u_c = 1 + r, \tag{5.12}$$

Table 1
Type of critical points for $\alpha^2 > 0$, also see Fig. 2

u:	0	2/3	$1+r$	order
$r > 0$	$\times_{\mathrm{t}}$	$\circ$	$\times$	$\times \circ \vert$
$0 > r > -1/3$	$\times_{\mathrm{c}}$	$\circ$	$\times_{\mathrm{p}}$	$\times \circ \vert$
$-1/3 > r > -1$	$\times_{\mathrm{c}}$	$\times_{\mathrm{c}}$	—	$\times \vert \times$
$-1 > r > -4/3$	$\circ$	$\times_{\mathrm{c}}$	$\times_{\mathrm{p}}$	$\vert \circ \times$
$r < -4/3$	$\circ$	$\times_{\mathrm{t}}$	$\times$	$\vert \circ \times$

($\circ$) points: stable, ($\times$) points: unstable. The subscript denotes the type of separatrix: t, travelling wave; p, periodic peakon, c, cuspon.

Table 2
Type of critical points for $\alpha^2 < 0$, also see Fig. 3

u:	0	2/3	$1-r$	order
$r > 1$	$\times_{\mathrm{ct}}$	$\circ$	—	$\vert \times \circ$
$1/3 < r < 1$	$\circ$	$\circ$	$\times_{\mathrm{pp}}$	$\circ \vert \circ$
$r < 1/3$	$\circ$	$\times_{\mathrm{tc}}$	—	$\times \circ \vert$

The index denotes the two (!) possible separatrices.

provided $3u_{\mathrm{c}}^2 - 2u_{\mathrm{c}} > 0$. These critical points only exist, provided either $r < -1$, or $r > -1/3$. Their critical value is $r(1+r)^2/2$. Stability of the critical points (5.11) is determined by the sign of the determinant of the Hessian of the energy. The determinant at theses critical points is

$$D(E_+) = (3u - 1)(r + 1 - u). \tag{5.13}$$

Varying r allows one to change the stability properties of the critical points (5.11) and the existence of the additional critical points (5.12), see Tables 1 and 2.

Fig. 2 shows typical pictures of the phase portrait in the phase plane u' versus u. In all pictures only some solutions that correspond to bounded travelling waves are shown in addition to all critical solutions. The top row shows the generic phase portraits, while in the bottom row the bifurcation values are illustrated.

Solutions corresponding to critical values of E are called peakons if they have a finite jump in first derivative of u and are called cuspons if the derivative at the jump diverges.

The main feature of Fig. 2 is a homoclinic orbit to the origin $u = 0$, $u' = 0$ with zero energy. The origin is unstable and the equilibrium $u' = 0$, $u = 2/3$ is stable. In Table 1, first row, the corresponding unstable point is denoted by ($\times$), the stable one by ($\circ$). The singular line is located at $u = u_{\mathrm{c}} = 1 + r$. In Fig. 2a the parameter r is large enough that the singular line $u = u_{\mathrm{c}}$ is well separated from the value $u = 1$ where the homoclinic orbit of the origin crosses the $u' = 0$-axis. Varying r so that the singular line $u = u_{\mathrm{c}}$ moves closer to the rightmost point $u = 1$ of the separatrix of the origin gives a travelling wave with a more and more pronounced peak. For the limiting case $r = 0$ which defines the peakon limit, the travelling wave shows a jump in the first derivative as shown in Fig. 2f. The travelling wave is now a peakon. One can vary r even further towards negative r to obtain the

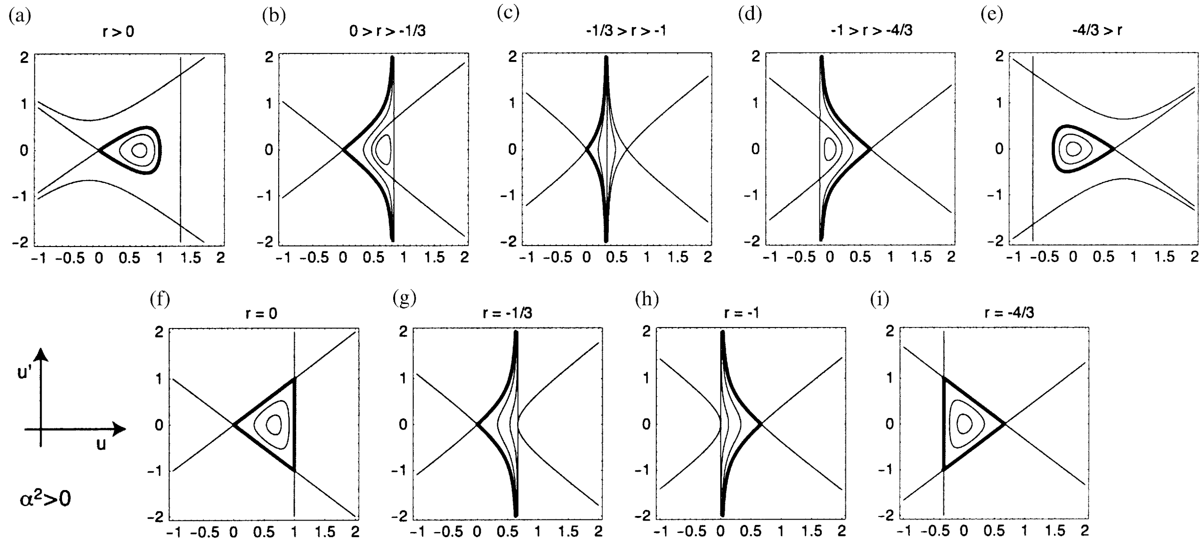

Fig. 2. Phase portraits for $\alpha^2 > 0$. There are 5 different cases (a)–(e) depending on the value of r, cmp. Table 1.

scenario depicted in Fig. 2b which allows for cuspons and periodic peakons. If $r > -1/3$ we have the two additional critical points (5.12) and if in addition $r < 0$ these critical points correspond to periodic peakons. The origin $u = 0$ which has $E = 0$ defines now a cuspon which has an infinite slope. For $-1 < r < -1/3$ the critical point at $u = 2/3$ becomes unstable and we have now two cuspons; one describing elevations and the other defining depressions. The typical phase portrait of this scenario is depicted in Fig. 2c. We can push the singular line u_c further across the u'-axis, by allowing $r < -1$ which is shown in Fig. 2d. Now we have a cuspon defining a depression wave and again periodic peakons corresponding to the additional critical points (5.12). The separatrix of these points reconnects if their energy equals the energy of the unstable saddle. Simple algebra shows that this is the case for $r = -4/3$. At this r-value the periodic peakons vanish and the cuspon degenerates into a peakon as depicted in Fig. 2i. For $r < -4/3$ a homoclinic orbit corresponding to smooth travelling waves of depression emerges as shown in Fig. 2e. We note here that the phase portraits in the original variables for these waves are reflected around the u'-axis when $\Delta < 0$. E.g. when $\Delta < 0$ and the Bond number is larger than σ_α, the case $r > 0$ is in fact a depression wave. Such scenarios for a similar, but different, equation have been studied in detail in Qian and Tang (2001). Another system which admits coexistence of regular solitary waves, peakons and cuspons has been studied in Grimshaw et al. (2002).

5.4. Case 2: $\alpha^2 < 0$

In Fig. 3 we show the different solution types for negative α^2. Again we show solutions that lead to compact (in u!) solutions of generic type in the top row and of special type in the bottom row. In addition to (5.11) there are two r dependent singular critical points for which

$$u' = \pm\sqrt{-(3u_c^2 - 2u_c)} \quad \text{and} \quad u = u_c = 1 - r, \tag{5.14}$$

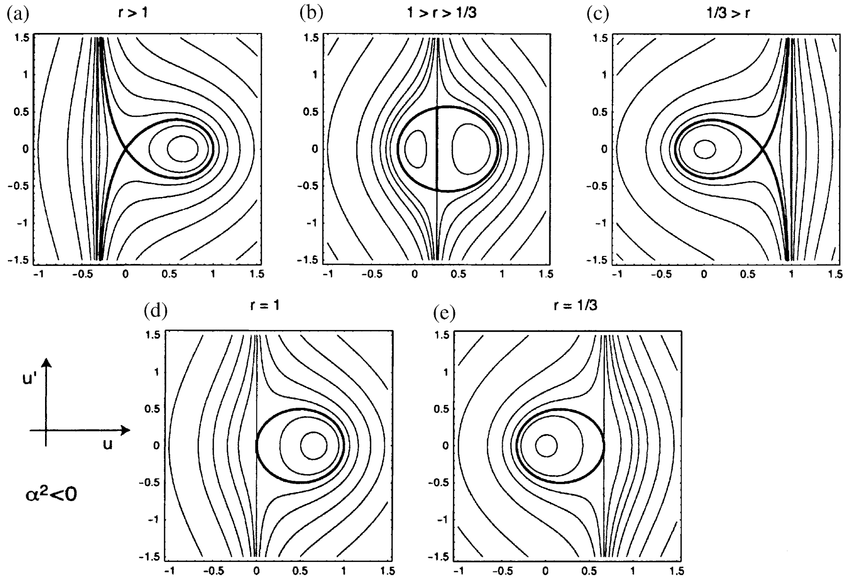

Fig. 3. Phase portraits for $\alpha^2 < 0$, cmp. Table 2.

provided $3u_c^2 - 2u_c < 0$. Therefore these critical points only exist if $1/3 < r < 1$. Their critical value is $r(1-r)^2/2$. The stability of the two r-independent critical points (5.11) is determined by the determinant of the Hessian

$$D(E_-) = (3u-1)(r-1+u). \tag{5.15}$$

Taking into account the condition for the existence of the additional critical points (5.14) we find that for $r > 1$ we have travelling waves described by the homoclinic orbit with energy $E = 0$. This is the regular solitary wave.

In Fig. 3d the singular line collides with the equilibrium and this gives rise to the compactons first found by Li et al. (1999). This happens again at $r = 1/3$, see Fig. 3e.

If $1/3 < r < 1$ both critical points (5.11) are stable and the additional critical points (5.14) exist only in this r-interval. The phase portrait Fig. 3b shows two periodic peakons, one being an elevation wave train, the other being a depression wave train. Further variation of r with $r < 1/3$ allows for depression travelling waves. We stress here that again the sign of Δ flips the pictures in the original variables.

5.5. Parameterisation of travelling waves

In Section 6, the physical relevance of these solution and their observability is investigated. We shall see that not all mathematically admissible solutions correspond to physically observable solutions.

The energy may be rewritten as

$$2E_\pm = \pm u'^2(u_c - u) + u^3 - u^2.$$

By an additional transformation of the independent variable $(\mathrm{d}s/\mathrm{d}\tau)^2 = u_c - u$ if $\alpha^2 > 0$, or $(\mathrm{d}s/\mathrm{d}\tau)^2 = u - u_c$ if $\alpha^2 < 0$ this can be reduced to the simple equation

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = \sqrt{u^3 - u^2 - 2E}. \tag{5.16}$$

This change of independent variable is called a Sundman transformation in the classical mechanics literature. For $E = 0$ and for $E = -2/27$, the right-hand side has double roots, so that the solution is given by (hyperbolic) trigonometric functions. Note that the parameter r is now completely hidden in the transformation from s to τ. Obviously this scaling fails for $u = u_c$, so that the solutions asymptotic to the singular equilibria have to be treated separately.

For $E = 0$ Eq. (5.16) has the well known solution

$$u(\tau) = \mathrm{sech}^2\, \tau/2. \tag{5.17}$$

Substituting this solution into the Sundman transformation leads to $\int (\pm(u_c - \mathrm{sech}^2\, \tau/2))^{1/2}\, \mathrm{d}\tau$. This becomes an elementary integral after the substitution $v = \sinh^2 \tau/2$. The scaled "time"-variable for $\alpha^2 > 0$ and $r > 0$ is

$$\frac{s(\tau)}{2} = \sqrt{u_c}\, \sinh^{-1}\left(\frac{\sinh \tau/2}{\sqrt{1 - 1/u_c}}\right) - \tanh^{-1}\left(\frac{\tanh \tau/2}{\sqrt{\tanh^2 \tau/2 + u_c - 1}}\right),$$

while for $\alpha^2 < 0$ and $r > 1$ it is

$$\frac{s(\tau)}{2} = \sqrt{-u_c}\, \sinh^{-1}\left(\frac{\sinh \tau/2}{\sqrt{1 - 1/u_c}}\right) + \tan^{-1}\left(\frac{\tanh \tau/2}{\sqrt{1 - u_c - \tanh^2 \tau/2}}\right).$$

Note that these expressions are still written down in the scaled coordinates. Going back to the original u we have to multiply the right-hand sides by Δ for u and by $|\alpha|$ for s. The curvature at the maximum is $-1/2r$. In the original variables we find $-\Delta^2/2\Gamma$ instead. The curvature depends essentially on σ, and diverges when $\sigma \to 1/3$, corresponding to $r \to 0$. The curvature is the same as for the ordinary KdV soliton. For $\alpha^2 < 0$ the curvature has the opposite sign.

The pulse-solutions (5.17) are solitary waves and inherit the properties such as elastic interaction by its two limiting equations. In fact, in Dullin et al. (2001) the spectral scattering problem is stated that allows an exact analytical treatment of initial value problems and wave interactions.[1] A typical picture of a collision of these solitary waves is shown in Fig. 4. We note that travelling waves with $c_0 \neq 0$ and $\Gamma \neq 0$ are very close to KdV solitary waves. The initial condition for this picture (at negative time not shown) was a Gaussian initial peak. After an elastic collision the only impact of the collision is a phase shift of the interacting waves. In principle this phase shift may be determined analytically by means of the inverse scattering technique. However, the Kodama transformation (3.1) used in Section 3, in particular the non-local term,

$$u_x \int^x u(x')\, \mathrm{d}x', \tag{5.18}$$

relates the phase shift of the different water wave equations to each other, i.e. equations (3.16), (4.3) and (2.19); Kodama (2001). This may be seen by noting that the integral in (5.18) is different after

[1] Note, in the formula after Eq. (5.5) in Dullin et al. (2001), the term $1/(4\alpha)$ should read $1/(4\alpha^2)$.

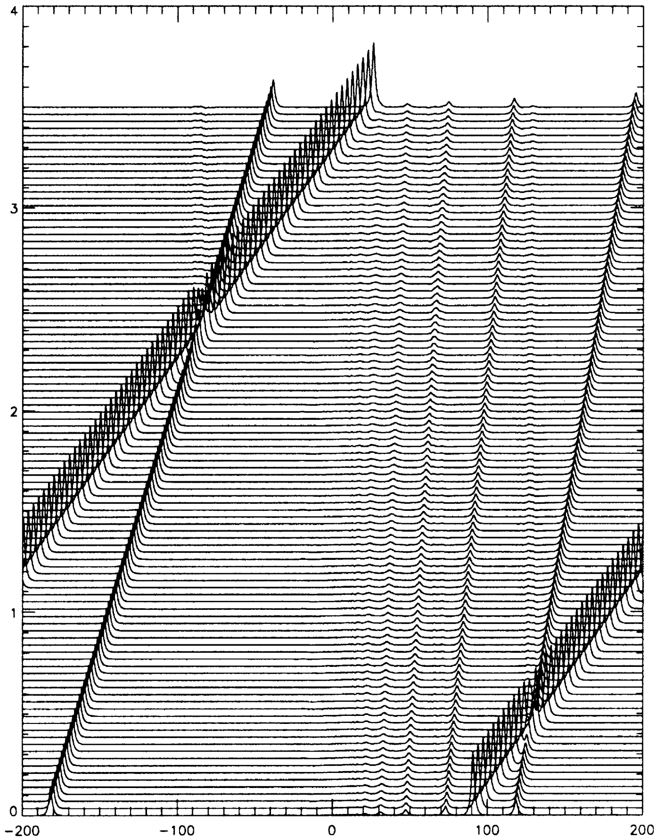

Fig. 4. Elastic collisions of solitary waves for $\alpha^2 = 1.5$, $\Gamma = 0.0115, c_0 = 0.001$.

the collision and hence keeps track of the waves during the collision. Let us assume two waves are about to collide. We integrate from negative spatial infinity to the position of the wave crest of the taller wave. The term (5.18) does not involve the second wave which is to the right of the integration boundary. However, after the collision, the taller wave is to the right of the smaller wave and in this case the integration domain in (5.18) from negative spatial infinity to the wave crest of the taller wave now includes the smaller wave, which provides information on the phase shift that occurs during the collision.

6. Physical relevance

How much of the richness of solutions discovered in Section 5 actually occurs in nature? One cannot claim that the solution types found in the travelling wave reduction are all present in the Euler equation. For example with non-vanishing σ any non-smoothness in the profile would be removed by the curvature term proportional to σ in (2.1). The following discussion should be understood in the sense that the profiles found are approximations to the true solutions.

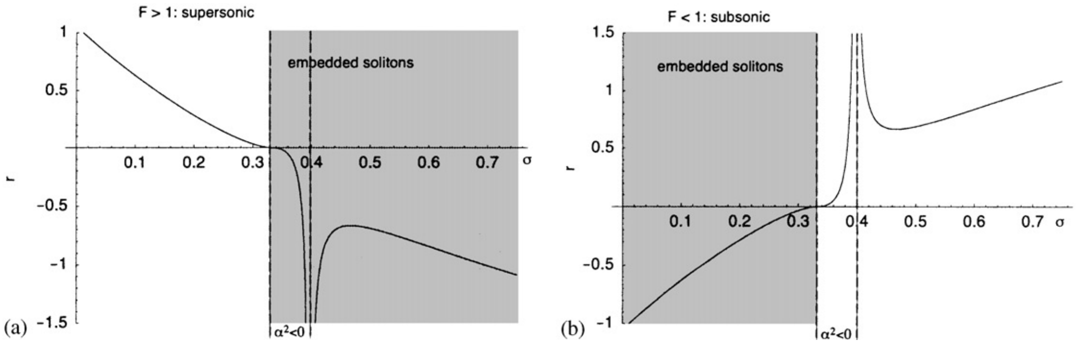

Fig. 5. The essential parameter r of (6.1) as a function of the Bond number σ for (a) $F > 1$ (supercritical, non-resonant for $\sigma < 1/3$) and (b) $F < 1$ (subcritical, non-resonant for $\sigma > 1/3$).

A discussion of the existence question may be phrased in terms of the two essential physical parameters, namely the Bond number σ and the Froude number F. In Fig. 1 we show typical dispersion relations for $\sigma < 1/3$ and $\sigma > 1/3$. We focus on solitary waves which are not inside the continuous spectrum (i.e. non-resonant, or non-embedded) for small k. Such solitary waves must be supercritical ($F > 1$) for $\sigma < 1/3$ and subcritical ($F < 1$) for $\sigma > 1/3$. This relation between the speed of the solitary wave and the surface tension is implicit in Eq. (3.16). The condition of non-resonance states that the solitary wave moves faster (slower) than any linear wave with small k.

We now investigate the dependence of the parameter r on the Bond number σ and the Froude number F. Using the expressions for α^2 and Γ (3.17)–(3.18) we find

$$r = \frac{\Gamma}{|\alpha|^2 \Delta} = \frac{10}{1-F}\,\frac{(1-3\sigma)^2}{45\sigma^2+30\sigma-19}\,\mathrm{sign}(\alpha^2). \tag{6.1}$$

The graph of this function is shown in Fig. 5. The σ-axis is separated into 3 different regions: The interval $(1/3, \sigma_\alpha)$ in the middle in which we have $\alpha^2 < 0$, corresponding to Fig. 3, and two other intervals, $[0, 1/3)$ and (σ_α, ∞), in which $\alpha^2 > 0$, corresponding to Fig. 2. Non-resonant travelling waves are found for $\sigma < 1/3$ if $F > 1$, hence for $r > 0$ in Fig. 5a, similarly for $\sigma > 1/3$ if $F < 1$, hence again for $r > 0$ in Fig. 5b. The physical significance of embedded solitons, which occur for $r < 0$ is not clear, but it might be interesting to investigate it. We conclude that $r > 0$ leaves only travelling waves of the type depicted in Fig. 2a or Fig. 3a. All other cases with $\alpha^2 > 0$ and $r < 0$ (Fig. 2b–e) are embedded solitons, which are beyond our present scope. It is pertinent to mention that $F < 1$ implies negative u (see (5.8)). Hence the travelling waves for $\sigma > 1/3$ are waves of depression whereas the waves for $\sigma < 1/3$ and $F < 1$ are elevations which is in accordance with the numerical observations for the full water wave system discussed above.

Remarkably, the restriction to $r > 0$ for $\sigma \in (1/3, \sigma_\alpha)$ still allows all cases shown in Fig. 3. Hence beside the travelling waves, there are periodic peakons, shelf waves, cuspons and compactons. For a given fluid, the Bond number σ is fixed and the Froude number F is a free parameter for the solution. For σ in the two outer intervals the only non-embedded solitons are those of Fig. 2a. If, however, $\sigma \in (1/3, \sigma_\alpha)$ changing F between 0 and 1 can produce the qualitatively different solutions of Fig. 3a–c. The standard soliton with $r > 1$ is always possible for F sufficiently close to 1. For

$\sigma \in (1/3, 1/9 + 4\sqrt{10}/45)$ periodic peakons are possible for sufficiently small F. For $\sigma \in (1/3, 5/21 + 4\sqrt{15}/105)$ also the shelf waves of Fig. 3c are possible for sufficiently small F. Since for σ in this interval viscous effects are presumably already large it is not clear whether any of these solution types are present in fluids. But one may still ask the question whether the full Euler equations would have similar solutions.

If $\alpha^2 < 0$ the Helmholtz operator is no longer invertible. However, this poses no problem for the derived equations. As a matter of fact the inverse transformation back to η does not involve the application of the inverse Helmholtz operator. Instead, it is achieved by simple substitution of the time derivatives and the Kodama-transformation along the lines of Section 4 where KdV5 is recovered from CH without the inversion of the Helmholtz operator. An issue here is the required accuracy of the inversion. Since the exotic solutions for $\alpha^2 < 0$ exist only for σ sufficiently close to 1/3 and the asymptotic expansion collapses for $\sigma = 1/3$, the predictions for this case may not be accurate. A study of the full Euler equations would be needed to resolve this issue. In this regard, we recall the investigations reported in Benjamin (1982) of anomalies in solitary shallow water wave behavior for $\sigma \simeq 1/3$.

We conclude that the peakon, which is the critical solution shown in Fig. 2f, apparently cannot exist in water waves. This is because the critical condition $r = 0$ implies $\Gamma = 0$, and considering (3.18) it follows that either the equilibrium depth vanishes, $h_0 = 0$, or the acceleration of gravity vanishes, $g = 0$. However, either of these conditions would invalidate the initial assumptions of the derivation. The only other way to achieve $\Gamma = 0$ is when $\sigma = 1/3$, but in this case v (and hence α^2) diverges.

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